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Linear algebra

Fall Semester 2013

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Problem Set #3

Due Tuesday 25 february in Class

Exercise 1 ($\star$)
Let V be a vector space and $W \subset V$ a subspace. Show that the relation

$$v \sim v' \Leftrightarrow v - v' \in W$$

defines an equivalence relation on V.

In the following two exercises, let $T : V \to W$ be a **linear map** between the vector spaces V and W that is: for any v_1, $v_2 \in V$ and $\lambda \in K$

(a) $T(\lambda \cdot v_1) = \lambda \cdot T(v_1)$;

(b) $T(v_1 + v_2) = T(v_1) + T(v_2)$

We denote 0_V (resp. 0_W) stands for the zero element of V (resp. W). Define the kernel of T to be the set

$$ker(T) := \{v \in V : T(v) = 0_W\}$$

and the range of T to be the set

$$Im(T) := \{T(v) : v \in V\}$$

Exercise 2 ($\star$)**:**

1. Prove that
   - $T(\sum_{i=1}^{n} \lambda_i v_i) = \sum_{i=1}^{n} \lambda_i T(v_i)$, for any $\lambda_i \in K$ and $v_i \in V$.
   - $T(0_V) = 0_W$.
   - $T(-v) = -T(v)$, for any $v \in V$;
2. Let $T_1 : V \to W$ and $T_2 : V \to W$ be linear maps and S be a spanning set of S. Suppose that $T_1(s) = T_2(s)$, for any $s \in S$. Prove that $T_1 = T_2$ (i.e. $T_1(v) = T_2(v)$, for any $v \in V$).
3. Let $dim_K(V) < \infty$, $\{v_1, ..., v_n\}$ a bases for V. Select any n vectors w_1, ..., w_n in W, prove that there is a unique linear map $T : V \to W$ such that $T(v_i) = w_i$, for any $1 \leq i \leq n$. (Be careful: Prove existence and unicity.)

4. Suppose that the linear map T is a bijection. Prove that its inverse T^{-1} is also a linear map. (Hint: $T \circ T^{-1} = Id_W$ and $T^{-1} \circ T = Id_V$. Recall that for any $w \in W$, $T^{-1}(w) =$ unique vector $v \in V$ such that $T(v) = w$.)

5. Prove also that if T is $\begin{cases} 1) \text{ injective} \\ 2) \text{ surjective} \\ 3) \text{ bijective} \end{cases}$, then T sends $\begin{cases} 1) \text{ independent sets} \\ 2) \text{ spanning sets} \\ 3) \text{ bases} \end{cases}$ of V into $\begin{cases} 1) \text{ independent sets} \\ 2) \text{ spanning sets} \\ 3) \text{ bases} \end{cases}$ of W

Exercise 3 ($\star$):

1. Prove that $ker(T)$ is a subspace of V and $Im(T)$ is a subspaces of W.

2. Prove that T is injective if and only if $Ker(T) = \{0\}$.

3. If $dim_K(V) < \infty$, Prove that $dim_K(Ker(T)) < \infty$, $dim_K(Im(T)) < \infty$ and that

$$dim_K(Ker(T)) + dim_K(Im(T)) = dim_K(V)$$

   (Hint : Get inspire by the proof of the course for $dim_K(V/W) = dim_K(V) + dim_K(W)$).

4. Suppose that $dim_K(V) = dim_K(W) < \infty$. Deduce from previous question that T is injective if and only if T is surjective if and only if T is bijective.

5. ($\star\star$ Bonus) For W a subspace of a vector space V, we have seen in class that the quotient map $\pi : V \to V/W$ is a linear map. Prove that:

   - $ker(\pi) = W$
   - $\pi : V \to V/W$ has the following universal property: given a vector space X and a linear map $T : V \to X$ such that $W \subset ker(T)$, there exists a unique linear map $\bar{T} : V/W \to X$ such that $\bar{T} \circ \pi = T$. (Hint: Notice that the last equality define already the map $\bar{T}$, be careful of making sure this map is well define, that is it does not depends on the choices of the representative and that it is unique).
     (*Note: It is common to say that* $\bar{T}$ *makes the diagram*

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\pi"'] & X \\ V/W \arrow[ur, "\bar{T}"'] & \end{tikzcd}$$

     commutes meaning that $\bar{T} \circ \pi = T$.)

[1]

[1] ($\star$) = easy , ($\star\star$)= medium, ($\star\star\star$)= challenge